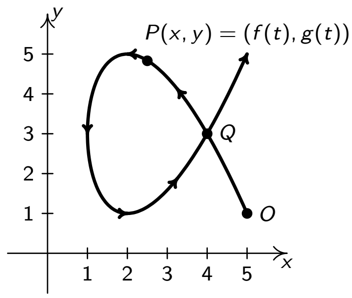
MATH 1700: SECTION 13.5: PARAMETRIC EQUATIONS

In this section, we introduce parametric equations - another method to algebraically describe curves in the plane. To motivate the idea, we imagine a bug crawling across a table top starting at the point O and tracing out a curve C in the plane, as shown below.



The curve C does not represent y as a function of x because it fails the Vertical Line Test and it does not represent x as a function of y because it fails the Horizontal Line Test. However, since the bug can be in only one *place* $P(x, y)$ at any given time t, we can define the x-coordinate of P as a function of t and the y-coordinate of P as another function of t. Traditionally, $f(t)$ is used for x and $g(t)$ is used for y.

The independent variable t in this case is called a **parameter** and the system of equations

$$\begin{cases} x &= f(t) \\ y &= g(t) \end{cases}$$

is called a **system of parametric equations** or a **parametrization** of the curve C. The parametrization of C endows it with an *orientation* and the arrows on C indicate motion in the direction of increasing values of t.

In this case, our bug starts at the point O, travels upwards to the left, then loops back around to cross its path at the point Q and finally heads off into the first quadrant.

It is important to note that the curve itself is a set of points and as such is devoid of any orientation. The parametrization determines the orientation and different parametrizations can determine different orientations.

If all of this seems hauntingly familiar, it should. By definition, the system of equations $\{x = \cos(t),\ y = \sin(t)$ parametrizes the Unit Circle, giving it a counter-clockwise orientation. More generally, the equations of circular motion $\{x = r\cos(\omega t),\ y = r\sin(\omega t)$ we developed earlier in the course are parametric equations which trace out a circle of radius r centered at the origin. If $\omega > 0$, the orientation is counter-clockwise; if $\omega < 0$, the orientation is clockwise. The angular frequency ω determines 'how fast' the object moves around the circle.

EXAMPLE 1: Sketch the curve described by $\left\{ \begin{array}{rcl} x & = & t^2 - 3 \\ y & = & 2t - 1 \end{array} \right.$ for $t \geq -2$.

The curve sketched out in Example 1 certainly looks like a parabola. Indeed if we use the technique of substitution on the system $\left\{x = t^2 - 3, \ y = 2t - 1\right.$ to eliminate the parameter t and get an equation involving just x and y.

To do so, we choose to solve the equation $y = 2t - 1$ for t to get $t = \frac{y+1}{2}$. Substituting this into the equation $x = t^2 - 3$ yields $x = \left(\frac{y+1}{2}\right)^2 - 3$ or, after some rearrangement, $(y+1)^2 = 4(x+3)$. The graph of this equation is a parabola with vertex $(-3, -1)$ which opens to the right, as required.

Technically speaking, the equation $(y + 1)^2 = 4(x + 3)$ describes the *entire* parabola, while the parametric equations $\left\{x = t^2 - 3, \ y = 2t - 1\right.$ for $t \geq -2$ describe only a *portion* of the parabola. In this case, we can remedy this situation by restricting the bounds on y. Since the portion of the parabola we want is exactly the part where $y \geq -5$, the equation $(y + 1)^2 = 4(x + 3)$ coupled with the restriction $y \geq -5$ describes the same curve as the given parametric equations. The one piece of information we can never recover after eliminating the parameter, however, is the orientation of the curve.

Eliminating the parameter and obtaining an equation in terms of x and y, whenever possible, can be a great help in graphing curves determined by parametric equations. If the system of parametric equations contains algebraic functions, as was the case in Example 1, then we can try the usual techniques of substitution and elimination on the system $\{x = f(t), \ y = g(t)$ to eliminate the parameter. If, on the other hand, the parametrization involves the trigonometric functions, the strategy changes slightly. In this case, it is often best to solve for the trigonometric functions and relate them using an identity.

EXAMPLE 2: Sketch the curves described by the following parametric equations.

1. $\begin{cases} x &= e^{-t} \\ y &= e^{-2t} \end{cases}$ for $t \geq 0$

2. $\begin{cases} x &= \sin(t) \\ y &= \csc(t) \end{cases}$ for $0 < t < \pi$

3. $\begin{cases} x &= 1 + 3\cos(t) \\ y &= 2\sin(t) \end{cases}$ for $0 \leq t \leq \frac{3\pi}{2}$

PARAMETRIZATIONS OF COMMON CURVES:

- The graph of $y = f(x)$ as x runs through some interval I is parametrized by:
  $\{x = t,\ y = f(t)$ as t runs through I.

- The graph of $x = g(y)$ as y runs through some interval I is parametrized by:
  $\{x = g(t),\ y = t$ as t runs through I.

- The graph of a directed line segment from (x_0, y_0) to (x_1, y_1) is parametrized by:
  $\{x = x_0 + (x_1 - x_0)t,\ y = y_0 + (y_1 - y_0)t$ for $0 \leq t \leq 1$.

- The graph of a circle or ellipse $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ where $a, b > 0$ is parametrized by:
  $\{x = h + a\cos(t),\ y = k + b\sin(t)$ for $0 \leq t < 2\pi$.

  NOTE: This will impart a *counter-clockwise* orientation.

One can verify the above formulas by eliminating the parameter and, when indicated, checking the orientation.

EXAMPLE 3: Find a parametrization for each of the following curves and check your answers.

1. $y = x^2$ from $x = -3$ to $x = 2$

2. $y = f^{-1}(x)$ where $f(x) = x^5 + 2x + 1$

3. The line segment which starts at $(2, -3)$ and ends at $(1, 5)$

4. The circle $x^2 + 2x + y^2 - 4y = 4$

5. The left half of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

ADJUSTING PARAMETRIC EQUATIONS:

- **Reversing Orientation:**

  Replacing every occurrence of t with $-t$ in a parametric description for a curve (including any inequalities which describe the bounds on t) reverses the orientation of the curve.

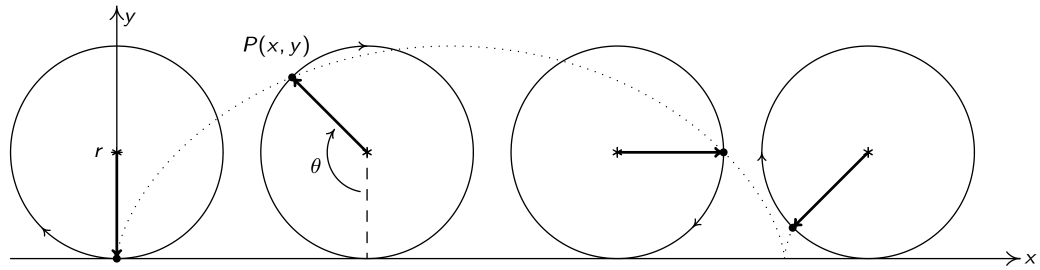
- **Shift of Parameter:**

  Replacing every occurrence of t with $(t-c)$ in a parametric description for a curve (including any inequalities which describe the bounds on t) shifts the start of the parameter t ahead by c units.

EXAMPLE 4: Adjust your parametrization for the left half of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that you found in Example 3 so that the path is traced out clockwise starting at $t = 0$.

THE CYCLOID:

Suppose a circle of radius r rolls along the positive x-axis at a constant velocity v as pictured below. Let θ be the angle in radians which measures the amount of clockwise rotation experienced by the radius:



Our goal is to find parametric equations for the coordinates of the point $P(x, y)$ in terms of θ. We first find parametric equations which describe the circular motion: clockwise starting from the bottom of the circle. Starting with the counter-clockwise parameterization of a circle of radius r which starts at the point $(r, 0)$, $\{x = r\cos(\theta),\ y = r\sin(\theta)$, we can reverse orientation and shift the parameter to get $\{x = -r\sin(\theta),\ y = -r\cos(\theta)$ as a clockwise parameterization which starts at the point $(0, -r)$.

Next, we adjust for the fact that the circle isn't stationary with center $(0, 0)$, but rather, is rolling along the positive x-axis. Since the velocity v is constant, we know that at time t, the center of the circle has traveled a distance vt down the positive x-axis. Furthermore, since the radius of the circle is r and the circle isn't moving vertically, we know that the center of the circle is always r units above the x-axis. Putting these two facts together, we have that at time t, the center of the circle is at the point (vt, r).

We know $v = \frac{r\theta}{t}$, or $vt = r\theta$. Hence, the center of the circle, in terms of the parameter θ, is $(r\theta, r)$. As a result, we need to modify the equations $\{x = -r\sin(\theta),\ y = -r\cos(\theta)$ by shifting the x-coordinate to the right $r\theta$ units (by adding $r\theta$ to the expression for x) and the y-coordinate up r units (by adding r to the expression for y).

We get $\{x = -r\sin(\theta) + r\theta,\ y = -r\cos(\theta) + r$, which can be written as $\{x = r(\theta - \sin(\theta)),\ y = r(1 - \cos(\theta))$. Since the motion starts at $\theta = 0$ and proceeds indefinitely, we set $\theta \geq 0$.

EXAMPLE 5: Find the parametric equations of a cycloid which results from a circle of radius 3 rolling down the positive x-axis as described above. Graph your answer using a graphing utility.

HOMEWORK: Section 13.5: Page 1197: 1 - 41 every other odd; 43, 44.